Dedicated to S.P. Novikov on the occasion of his 60-th birthday

LUSTERNIK - SCHNIRELMAN THEORY FOR CLOSED 1-FORMS

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Abstract. S. P. Novikov developed an analog of the Morse theory for closed 1-forms. In this paper we suggest an analog of the Lusternik - Schnirelman theory for closed 1-forms. For any cohomology class $\xi \in H^1(X, \mathbf{R})$ we define an integer $\mathrm{cl}(\xi)$ (*the cup-length associated with* ξ); we prove that any closed 1-form representing ξ has at least $\mathrm{cl}(\xi) - 1$ critical points. The number $\mathrm{cl}(\xi)$ is defined using cup-products in cohomology of some flat line bundles, such that their monodromy is described by complex numbers, which are not Dirichlet units.

§1. The main result

1.1. Let X be a closed manifold and let $\xi \in H^1(X;\mathbf{Z})$ be a nonzero cohomology class. The Novikov inequalities [N] estimate the numbers of critical points $c_i(\omega)$ of different indices of any closed 1-form ω with Morse singularities on X lying in the class ξ.

Novikov type inequalities were constructed in [BF1] for closed 1-forms with slightly more general singularities (non-degenerate in the sense of Bott [B]). In [BF2] an equivariant generalization of the Novikov inequalities was found.

In this paper we will consider the problem of estimating the number of critical points of closed 1-forms ω with no non-degeneracy assumption. We suggest here a version of the Lusternik - Schnirelman theory for closed 1-forms.

We will define (cf. 1.2 below) a nonnegative integer $\mathrm{cl}(\xi)$, which we will call *the cup-length associated with* ξ. It is defined in terms of cup-products of some local systems constructed using ξ.

The main result of the paper consists of the following:

Theorem 1. *Let ω be a closed 1-form on X lying in an integral cohomology class $\xi \in H^1(X;\mathbf{Z})$. Let $S(\omega)$ denote the set of critical points of ω, i.e. the set of points $p \in X$ such that $\omega_p = 0$. Then the Lusternik - Schnirelman category of $S(\omega)$ satisfies*

$$\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(\xi) - 1. \tag{1-1}$$

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In particular, if the set of critical points $S(\omega)$ is finite then for the total number $|S(\omega)|$ of the critical points,

$$|S(\omega)| \geq \mathrm{cl}(\xi) - 1. \tag{1-2}$$

Here $\mathrm{cat}(S)$ denotes the classical Lusternik - Schnirelman category of $S = S(\omega)$, i.e. the least number k, so that S can be covered by k closed subsets $A_1 \cup \dots \cup A_k$ such that each inclusion $A_j \to S$ is null-homotopic.

A proof of Theorem 1 is given in §2.

1.2. The cup-length $\mathrm{cl}(\xi)$**.** Let $\xi \in H^1(X;\mathbf{Z})$ be an integral cohomology class. For any nonzero complex number $a \in \mathbf{C}^*$ denote by $\mathcal{E}_a \to X$ the complex flat line bundle determined by the following condition: the monodromy along any loop $\gamma \in \pi_1(X)$ is the multiplication by $a^{\langle \xi, \gamma\rangle} \in \mathbf{C}$. If $a, b \in \mathbf{C}^*$ we have the canonical isomorphism of flat line bundles $\mathcal{E}_a \otimes \mathcal{E}_b \simeq \mathcal{E}_{ab}$. Therefore we have the cup-product

$$\cup : H^i(X;\mathcal{E}_a) \otimes H^j(X;\mathcal{E}_b) \to H^{i+j}(X;\mathcal{E}_{ab}). \tag{1-3}$$

Definition. *The cup-length* $\mathrm{cl}(\xi)$ *is the largest integer k such that there exists a nontrivial k-fold cup product*

$$H^{d_1}(X;\mathcal{E}_{a_1}) \otimes H^{d_2}(X;\mathcal{E}_{a_2}) \otimes \dots \otimes H^{d_k}(X;\mathcal{E}_{a_k}) \to H^d(X;\mathcal{E}_a), \tag{1-4}$$

where $d = d_1 + \dots + d_k$, $d_1 > 0, d_2 > 0, \dots, d_k > 0$, $a = a_1 a_2 \dots a_k$ and among the complex numbers $a_1, \dots, a_k \in \mathbf{C}^*$ at least two are not Dirichlet units.

Recall that *a Dirichlet unit* is defined as a complex number $b \neq 0$ such that b and its inverse b^{-1} are algebraic integers. In other words, Dirichlet units can be characterized as roots of polynomial equations

$$\gamma_0 b^n + \gamma_1 b^{n-1} + \dots + \gamma_{n-1} b + \gamma_n = 0,$$

where all γ_i are integers and $\gamma_0 = \pm 1, \gamma_n = \pm 1$.

Note that the cup-length $\mathrm{cl}(\xi)$ satisfies

$$0 \leq \mathrm{cl}(\xi) \leq \dim X. \tag{1-5}$$

In 1.8 we will see examples showing that $\mathrm{cl}(\xi) = \dim X$ is possible.

From Theorem 1 we obtain the following simple corollary:

1.3. Corollary. *Assume that X is a closed manifold, $\xi \in H^1(X;\mathbf{Z})$, and for some $a \in \mathbf{C}^*$, which is not a Dirichlet unit, $H^*(X;\mathcal{E}_a) \neq 0$. Then any closed 1-form ω on X in class ξ has at least one critical point.*

Proof 1. Note that $\mathrm{cl}(\xi) \geq 2$ if $\mathrm{cl}(\xi) > 0$. Indeed, assume that some $H^i(X;\mathcal{E}_a)$ is non-trivial with $a \in \mathbf{C}^*$ not a Dirichlet unit. Then by the Poincaré duality there is a non-trivial product $H^i(X;\mathcal{E}_a) \otimes H^{n-i}(X;\mathcal{E}_{a^{-1}}) \to H^n(X;\mathbf{C})$ (where $n = \dim X$), and so we obtain $\mathrm{cl}(\xi) \geq 2$. Hence by Theorem 1, $|S(\omega)| \geq 1$. $\square$

Proof 2. Corollary 1.3 has also a brief proof independent of Theorem 1, which also explains why the definition of the cup-length $\mathrm{cl}(\xi)$ requires that at least two of the numbers a_i in (1-4), describing the monodromy, are not Dirichlet units. Namely,

suppose that $\xi \neq 0$ (for $\xi = 0$ the form ω is a function and the statement is trivial) and there exists a closed 1-form ω in class ξ having no critical points. Construct a smooth map $f : X \to S^1$ with $\omega = f^*(d\theta)$, where $d\theta$ is the angular form on the circle. Here

$$f(x) = \int_{x_0}^{x} \omega \qquad \mod \mathbf{Z}$$

(we assume that ξ is indivisible). If ω has no critical points then f is a fibration. Therefore, X is the mapping torus of a diffeomorphism $h : F \to F$, where F is the fiber of f. Hence we obtain (using the Mayer - Vietoris sequence)

$$\begin{aligned} H^i(X; \mathcal{E}_a) \simeq \ker[h^* - a : H^i(F; \mathbf{C}) \to H^i(F; \mathbf{C})] \oplus \\ \operatorname{coker}[h^* - a : H^{i-1}(F; \mathbf{C}) \to H^{i-1}(F; \mathbf{C})] \end{aligned}$$

and now our statement follows from the obvious fact that *any eigenvalue of a diffeomorphism of a compact manifold, acting on the cohomology, is a Dirichlet unit.* $\square$

From this argument it is clear that Theorem 1 becomes false if we allow the numbers a_i in the definition of the cup-length in 1.2 to be Dirichlet units. Indeed, one may construct mapping tori X, which admit closed 1-forms with no critical points and may have arbitrarily long non-trivial products (1-4) with a_i Dirichlet units (corresponding to the eigenvalues of the monodromy).

1.4. Relation to the Novikov numbers. The following theorem (which is essentially known and stated here only for the sake of completeness) describes the relation between the cohomology $H^i(X; \mathcal{E}_a)$ and *the Novikov numbers* $b_i(\xi)$, associated with a cohomology class $\xi \in H^1(X; \mathbf{Z})$.

Theorem 2. *Let X be a closed manifold and let $\xi \in H^1(X; \mathbf{Z})$ be an integral cohomology class. For fixed q the function $a \mapsto \dim_{\mathbf{C}} H^q(X; \mathcal{E}_a)$, $\quad a \in \mathbf{C}^*$ has the following behavior:*

(a) it is constant except at finitely many jump points $a = a_1, \dots, a_N$;

(b) the common value of $\dim_{\mathbf{C}} H^q(X; \mathcal{E}_a)$ for $a \neq a_1, \dots, a_N$ equals the Novikov number $b_q(\xi)$;

(c) for $a \in \mathbf{C}^*$, being one of the jump points $a_1, \dots, a_N$, the dimension of the cohomology $\dim_{\mathbf{C}} H^q(X; \mathcal{E}_a)$ is greater than the Novikov number $b_q(\xi)$;

(d) the jump points $a_1, \dots, a_N$ are algebraic numbers (not necessarily algebraic integers).

All statements of Theorem 2 except the last one, were announced in [N3] (even in a more general form). We will give a simple independent proof in §3.

We point out here that as it follows from Theorem 2 *for a transcendental $a \in C$, the dimension of the vector space $H^i(X; \mathcal{E}_a)$ does not depend on a and equals the Novikov number $b_i(\xi)$.*

1.5. Remarks. 1. A crude estimate for the cup-length $\operatorname{cl}(\xi)$ can be obtained by taking the maximal length of a non-trivial product (1-4) with $a_1, \dots, a_k$ *transcendental.* We will give an example (cf. 1.8, example 3) showing that this estimate can be really worse than the one provided by Theorem 1.

2. *In the longest nontrivial product (1-4) the number a must be equal 1 and the number d must be equal the dimension of the manifold* $\dim X$. Indeed, suppose that we have a nontrivial product $v_1 \cup \cdots \cup v_k \in H^d(X; \mathcal{E}_a)$ with $a = a_1 \ldots a_k$ not equal 1 or with $d < \dim X$, and at least two among the numbers $a_1, \ldots, a_k$ are not Dirichlet units. Then $a \neq 1$ implies $d < \dim X$ and (using the Poincaré duality) we may find a class $w \in H^{\dim X - d}(X; \mathcal{E}_{a^{-1}})$ with $v_1 \cup \cdots \cup v_k \cup w \neq 0$. Hence $\mathrm{cl}(\xi) > k$. $\square$

1.6. Forms with non-integral periods. In general, the cohomology class determined by a closed 1-form ω belongs to $H^1(X, \mathbf{R})$, i.e. it has real coefficients. It is clear that multiplying ω by a non-zero constant $\lambda \neq 0$ does not change the set of critical points $S(\omega)$ and multiplies the cohomology class by λ. Hence Theorem 1 also gives estimates in the case of *cohomology classes* $\xi \in H^1(X, \mathbf{R})$ *of rank 1* (i.e. for classes, which are real multiples of integral classes) if we define the associated cup-length $\mathrm{cl}(\xi)$ as follows

$$\mathrm{cl}(\lambda\xi) = \mathrm{cl}(\xi), \quad \lambda \in \mathbf{R}, \ \lambda \neq 0, \quad \xi \in H^1(X, \mathbf{Z}).$$

Recall, that given a cohomology class $\xi \in H^1(X, \mathbf{R})$, its *rank* is defined as the rank of the abelian group, which is the image of the homomorphism $H_1(X, \mathbf{Z}) \to \mathbf{R}$, determined by ξ. Note that the cohomology classes of rank 1 are dense in $H^1(X, \mathbf{R})$. Therefore the following definition makes sense.

Definition. Given a class $\xi \in H^1(X, \mathbf{R})$ of rank > 1, we define $\mathrm{cl}(\xi)$ as the largest number k, such that there exists a sequence of rank 1 classes $\xi_m \in H^1(X, \mathbf{R})$ with

$$\mathrm{cl}(\xi_m) \geq k, \qquad \lim_{m \to \infty} \xi_m = \xi, \tag{1-7}$$

and each ξ_m, considered as a homomorphism $H_1(M; \mathbf{Z}) \to \mathbf{R}$, vanishes on the kernel of the homomorphism $\xi : H_1(M; \mathbf{Z}) \to \mathbf{R}$.

Theorem 3. *Let ω be a closed 1-form on X lying in a cohomology class $\xi \in H^1(X; \mathbf{R})$. Let $S(\omega)$ denote the set of critical points of ω. Then the Lusternik - Schnirelman category of $S(\omega)$ satisfies*

$$\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(\xi) - 1. \tag{1-8}$$

In particular, if the set of critical points $S(\omega)$ is finite then for the total number $|S(\omega)|$ of the critical points,

$$|S(\omega)| \geq \mathrm{cl}(\xi) - 1. \tag{1-9}$$

For the proof see §3.

1.7. Connected sums. Let X_1 and X_2 be two closed n-dimensional manifolds. We will denote by $X_1 \# X_2$ their connected sum. Given cohomology classes $\xi_\nu \in H^1(X_\nu; \mathbf{R})$, where $\nu = 1, 2$, the class $\xi_1 \# \xi_2 \in H^1(X_1 \# X_2; \mathbf{R})$ is well defined (in an obvious way).

In the description of examples (cf. 1.8) we will use the following statement:

Proposition 1.

$$\mathrm{cl}(\xi_1 \# \xi_2) = \max\{\mathrm{cl}(\xi_1), \mathrm{cl}(\xi_2)\}. \tag{1-10}$$

Proof is given in §3.

1.8. Examples. 1. In the notations of the previous subsection, let $\xi_1 = 0$ and suppose that $\xi_2 \neq 0$ can be realized by a closed 1-from with no critical points (for example, fibration over the circle). Then we obtain from Proposition 1 that $\mathrm{cl}(\xi_1 \# \xi_2) = \mathrm{cl}(\xi_1)$. Since $\xi_1 = 0$, the cup-length $\mathrm{cl}(\xi_1)$ is the usual cup-length of the manifold X_1 with rational coefficients.

To have a specific example, let us take $X_1 = T^n$, $X_2 = S^1 \times S^{n-1}$, $\xi_1 = 0$ and $\xi_2 \in H^1(X_2; \mathbf{Z})$ being a generator. Then we have for $\xi = \xi_1 \# \xi_2 \in H^1(X_1 \# X_2; \mathbf{R})$

$$\mathrm{cl}(\xi_1 \# \xi_2) = n. \tag{1-11}$$

Therefore, by Theorem 1, any closed 1-form ω on $X_1 \# X_2$ lying in class ξ has a least $n-1$ critical points.

2. In a similar way one may construct examples of cohomology classes of higher rank with many critical points. Namely, suppose that $X_1 = T^n$ and $\xi_1 = 0$; take for X_2 arbitrary closed manifold of dimension n with a cohomology class $\xi_2 \in H^1(X_2; \mathbf{R})$ of rank q. Then for the class $\xi = \xi_1 \# \xi_2 \in H^1(X_1 \# X_2; \mathbf{R})$ (having rank q) we again obtain $\mathrm{cl}(\xi) = n$ (by comparing Proposition 1 with (1-5).

One may take, for example, $X_2 = T^q \times S^{n-q}$ with ξ_2 induced from a maximally irrational class on the torus T^q.

3. Let X be a 3-dimensional manifold obtained by 0-framed surgery on the knot 5_2:

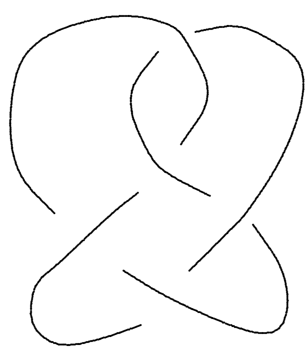

Figure 1.

This knot has Alexander polynomial $\Delta(\tau) = 2 - 3\tau + 2\tau^2$. Then $H^1(X; \mathbf{Z}) = \mathbf{Z}$ and taking $\xi \in H^1(X; \mathbf{Z})$ to be a generator we find that $H^1(X; \mathcal{E}_a)$ is trivial for all $a \in \mathbf{C}^*$, which are not the roots of the Alexander polynomial. It is easy to check that if a is one of the roots of $2 - 3a + 2a^2 = 0$ then $H^1(X; \mathcal{E}_a) \neq 0$. Note that the roots of $2 - 3a + 2a^2 = 0$ are not Dirichlet units. Hence we obtain (using Theorem 2) that all Novikov numbers are trivial, however by Corollary 1.3 we obtain that any closed 1-forms in the class ξ has at least 1 critical point.

4. Let X_g be a compact Riemann surface of genus $g > 1$. Then for any $\xi \in H^1(X_g; \mathbf{Z})$ and $a \neq 1$ holds $\dim H^1(X_g; \mathcal{E}_a) = 2g - 2$ and all other cohomology groups are trivial. Then we obtain that $\mathrm{cl}(\xi) = 2$ (cf. Proof 1 of Corollary 1.3). Thus Theorem 1 predicts existence of one critical point of any closed 1-form in any non-trivial cohomology class (which also follows from the Hopf's theorem).

We observe that *for any $\xi \in H^1(X_g; \mathbf{Z})$ with $\xi \neq 0$, $g > 1$ there exists a closed 1-form on the surface X_g lying in class ξ and having precisely one critical point.*

Indeed, it is known that on the surface X_{g-1} of genus $g-1$ there exists a smooth function with precisely three critical points: a maximum, a minimum and a saddle point, cf. [DNF], chapter 2, figure 81. Let $f : X_{g-1} \to \mathbf{R}$ be such function. We may assume that the critical values of f are $-1, 0, 1$. The level sets $f^{-1}(-1/2) = C_{-1}$ and $f^{-1}(1/2) = C_1$ are circles. If one picks an orientation of the surface X_{g-1}, these circles become oriented; if the orientation of X_{g-1} is reversed then the orientations of C_{-1} and C_1 are reversed. Let now X_g be obtained from $f^{-1}([-1/2, 1/2])$ by identifying the points of the circles C_{-1} and C_1 by a diffeomorphism $C_{-1} \to C_1$ preserving the orientations (the meaning of this is clear from the remark above). We obtain a map $h : X_g \to S^1 = \mathbf{R}/\mathbf{Z}$ by $h(x) = f(x) \mod \mathbf{Z}$ for $x \in X_g$. This gives a map to the circle S^1 with precisely one critical point. Clearly, the closed 1-form $h^*(d\theta)$ has only one critical point.

Hence our estimates are exact in the case of surfaces.

§2. Proof of Theorem 1

2.1. Since we assume that the cohomology class ξ of ω is integral, $\xi \in H^1(X, \mathbf{Z})$, there is a smooth map $f : X \to S^1$ such that $\omega = f^*(d\theta)$, where $d\theta$ is the standard angular form on the circle $S^1 \subset \mathbf{C}$, $S^1 = \{z; |z| = 1\}$. We will also assume (without loss of generality) that ξ is indivisible and that $1 \in S^1$ is a regular value of f.

Denote $f^{-1}(1)$ by $V \subset X$; it is a smooth codimension one submanifold. Let N denote the manifold obtained by cutting of X along V. We will denote by $\partial_+ N$ and $\partial_- N$ the components of the boundary of N. We get a smooth function

$$g : N \to [0, 1], \tag{2-1}$$

so that

(1) g is identically 0 on $\partial_+ N$ and identically 1 on $\partial_- N$ and has no critical points near ∂N;
(2) letting $p : N \to X$ denote the natural identification map, then for any $x \in N$,
$$f(p(x)) = \exp(2\pi i g(x));$$
(3) p maps the set $S(g)$ of critical points of g homeomorphically onto the set $S(\omega)$ of critical points of ω.

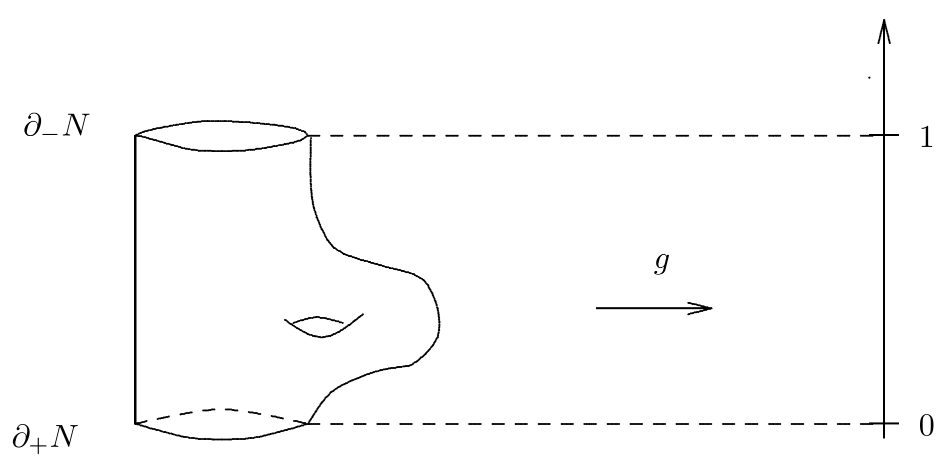

Figure 2

2.2. For any subset $X \subset N$ containing $\partial_+ N$ we will denote by $\mathrm{cat}_{(N,\partial_+N)}(X)$ the minimal number k such that X can be covered by $k+1$ closed subsets

$$X \subset A_0 \cup A_1 \cup A_2 \cup \dots \cup A_k \subset N$$

with the following properties:

(1) A_0 is *a collar of* $\partial_+ N$, i.e. it contains a neighborhood of $\partial_+ N$ and the inclusion $A_0 \to N$ is homotopic to a map $A_0 \to \partial_+ N$ keeping the points of $\partial_+ N \subset A$ fixed;

(2) for $j = 1, 2, \dots, k$, each set A_j is disjoint from $\partial_+ N$ and the inclusion $A_j \to N$ is null-homotopic.

The number $\mathrm{cat}_{(N,\partial_+N)}(X)$ can be viewed as a relative version of the Lusternik - Schnirelman category.

Our purpose in this subsection is to prove the inequality

$$\mathrm{cat}\, S(\omega) \geq \mathrm{cat}_{(N,\partial_+N)}(N). \tag{2-2}$$

The arguments here are modifications of the standard arguments.

We will need Lemmas 1 - 4.

Lemma 1. *Let $X, Y \subset N$ be two closed subsets containing a neighborhood of $\partial_+ N$. Suppose that there exists a deformation $G_t : Y \to N$, $t \in [0,1]$, such that $G_0 =$ inclusion $: Y \to N$, $G_1(Y) \subset X$ and $G_t(x) = x$ for all $x \in \partial_+ N$, $t \in [0,1]$. Then*

$$\mathrm{cat}_{(N,\partial_+N)}(Y) \leq \mathrm{cat}_{(N,\partial_+N)}(X). \tag{2-3}$$

Proof. Suppose that $\mathrm{cat}_{(N,\partial_+N)}(X) = k$ and let $A_0 \cup A_1 \cup \dots \cup A_k$ be a cover of X by closed subsets as above. Set $B_0 = G_1^{-1}(A_0)$ and for $j = 1, 2, \dots, k$ let B_j be defined as $G_1^{-1}(A_j)$, with a small cylindrical neighborhood of $\partial_+ N$ removed. Let us show that the sets B_j, $j = 0, 1, 2, \dots, k$ satisfy the requirements of the above definition. We have the following deformation $G_t|_{B_j} : B_j \to N$, $t \in [0,1]$, which starts with the inclusion $B_j \to N$ and ends with a map $B_j \to A_j$. After that for $j > 0$ we may apply the deformation which shrinks A_j to a point; for $j = 0$ we apply the deformation which brings A_0 to $\partial_+ N$ keeping $\partial_+ N \subset A_0$ fixed. This gives a covering of Y with the required properties. Therefore, $\mathrm{cat}_{(N,\partial_+N)}(Y) \leq k$. $\square$

Lemma 2. *(a) Let $A \subset N - \partial_+ N$ be a compact subset such that the inclusion $A \to N$ is null-homotopic. Then for any $\epsilon > 0$ small enough the ϵ-neighborhood $U_\epsilon(A)$ of A is also null-homotopic in N. (b) Let A be a closed subset containing a neighborhood of $\partial_+ N$, such that A can be deformed into $\partial_+ N$ in N keeping the points of $\partial_+ N$ fixed. Then for small ϵ the ϵ-neighborhood $U_\epsilon(A)$ of A can also be deformed into $\partial_+ N$ keeping the points of $\partial_+ N$ fixed.*

Proof. Let us prove (a); the proof for (b) is the same. Suppose that $G : A \times [0,1] \to N$ is the given deformation of A into $\partial_+ N$. Embed N into some Euclidean space $\mathbf{R}^n$. G defines a mapping from the closed subset $A \times [0,1] \subset N \times [0,1]$ to $\mathbf{R}^n$. By the Tietze theorem there exists a continuous map $\tilde{G} : N \times [0,1] \to \mathbf{R}^n$ which coincides with G on $A \times [0,1]$. Let U be a small tubular neighborhood of N in $\mathbf{R}^n$ and let

$\pi : U \to N$ be the projection (retraction). Suppose that ϵ is so small that the ϵ-neighborhood of $A \times [0,1]$ is contained in $\tilde{G}^{-1}(U)$. Then the image $\tilde{G}(U_\epsilon(A \times [0,1]))$ is contained in U and, composing $\tilde{G}$ with the projection π, we obtain a deformation $U_\epsilon(A) \times [0,1] \to N$. The final map of this deformation brings A into $\partial_+ N$ and so the rest of $U_\epsilon(A)$ is contained in a collar of $\partial_+ N$. Hence $\tilde{G}(U_\epsilon(A \times 1))$ could be brought into $\partial_+ N$ by another deformation. □

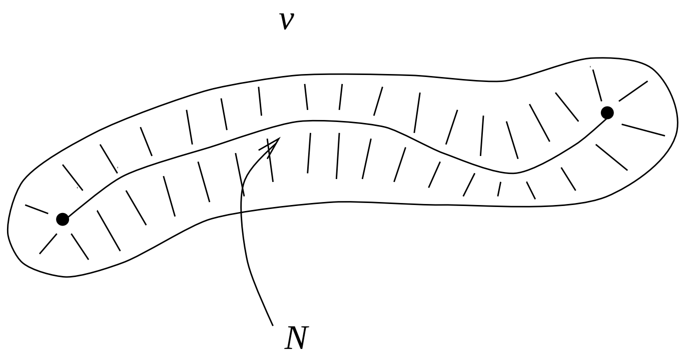

Figure 3

Lemma 3. *For any closed subset $X \subset N$ containing $\partial_+ N$ there exists $\epsilon > 0$ such that*

$$\mathrm{cat}_{(N,\partial_+ N)}(U_\epsilon(X)) = \mathrm{cat}_{(N,\partial_+ N)}(X). \tag{2-4}$$

Proof. The inequality $\mathrm{cat}_{(N,\partial_+ N)}(U_\epsilon(X)) \geq \mathrm{cat}_{(N,\partial_+ N)}(X)$ is obvious. The opposite inequality follows from Lemma 2. □

Lemma 4. *For a pair of closed subsets $X, Y \subset N$ such that X contains $\partial_+ N$, it holds*

$$\mathrm{cat}_{(N,\partial_+ N)}(X) + \mathrm{cat}_N(Y) \geq \mathrm{cat}_{(N,\partial_+ N)}(X \cup Y). \tag{2-5}$$

Here $\mathrm{cat}_N(Y)$ denotes the Lusternik-Schnirelman category of Y with respect to N, i.e. the minimal numbers k such that Y can be covered by k closed subsets which are null-homotopic in N.

Proof. Obvious. □

We are in a position now *to prove the inequality* (2-2).

We may assume that $S(\omega) = S(g)$ has only finitely many connected components, since otherwise $\mathrm{cat}(S(\omega))$ is infinite, and Theorem 1 is obviously true. Also, the function g (defined in 2.1) is constant on each connected component of the critical point set $S(g)$ (this follows from the Sard's theorem since the image of a connected component under g must be connected and must have measure zero). Hence we obtain that the set of critical values of function g is finite.

Consider the function

$$F(\mu) = \mathrm{cat}_{(N_\mu,\partial_+ N)}(N_\mu), \quad \text{where} \quad N_\mu = g^{-1}([0,\mu]), \qquad \mu \in [0,1].$$

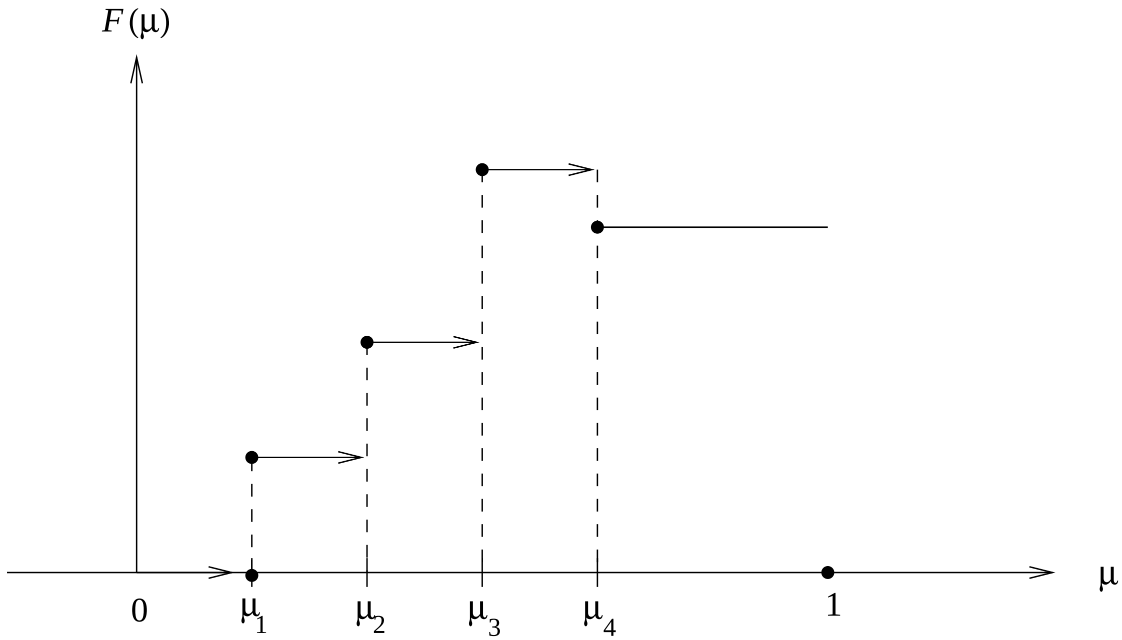

Figure 4

It is clear that $F(\mu) = 0$ for $\mu \geq 0$ small, and $F(\mu) = \mathrm{cat}_{(N,\partial_+ N)}(N)$ for $\mu \leq 1$ close to 1. Moreover, from the basic theorems of the Morse theory it follows that $F(\mu)$ is constant on each segment $[\mu, \mu']$, containing no critical values. Thus, $F(\mu)$ is a step function which may have finitely many jumps $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_s$ and all the jump points $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_s$ are the critical values of the function g.

We want to show that for any critical value $\mu \in [0, 1]$,

$$F(\mu) - F(\mu - \epsilon) \leq \mathrm{cat}(S_\mu), \tag{2-6}$$

where S_μ is the set of all critical points of g on the level $g^{-1}(\mu)$. Here $\epsilon > 0$ is so small that $[\mu - \epsilon, \mu)$ contains no critical values. To prove (2-6) we observe that choosing $\delta > 0$ small enough we get (by Lemma 4)

$$\mathrm{cat}_{(N_\mu,\partial_+ N)}(N_\mu - U_\delta(S_\mu)) + \mathrm{cat}_{N_\mu}(\overline{U}_\delta(S_\mu)) \geq F(\mu) \tag{2-7}$$

and also by Lemma 1 (using the deformation determined by the gradient flow of the function g)

$$\begin{aligned}&\mathrm{cat}_{(N_\mu,\partial_+ N)}(N_\mu - U_\delta(S_\mu)) \leq \mathrm{cat}_{(N_\mu,\partial_+ N)}(N_{\mu-\epsilon}) \leq \\ &\mathrm{cat}_{(N_{\mu-\epsilon},\partial_+ N)}\, N_{\mu-\epsilon} = F(\mu - \epsilon).\end{aligned} \tag{2-8}$$

In addition we have

$$\mathrm{cat}_{N_\mu}(\overline{U}_\delta(S_\mu)) = \mathrm{cat}_{N_\mu}(S_\mu) \leq \mathrm{cat}(S_\mu). \tag{2-9}$$

Combining (2-7), (2-8) and (2-9) proves (2-6).

The total jump of the function F on the interval $[0,1]$ equals $\mathrm{cat}_{(N,\partial_+N)}(N)$ and by (2-6) at each critical value μ the value of the jump does not exceed the Lusternik - Schnirelman category $\mathrm{cat}(S_\mu)$ of the set of critical points on the level. (Note that $F(\mu)$ may also have some negative jumps.) Hence the total jump $\mathrm{cat}_{(N,\partial_+N)}(N)$ does not exceed

$$\sum_\mu \mathrm{cat}(S_\mu) = \mathrm{cat}(S(g)) = \mathrm{cat}(S(\omega)). \tag{2-10}$$

This proves (2-2).

2.3. The deformation complex. Later (cf. 2.7) we will prove that

$$\mathrm{cat}_{(N,\partial_+N)}(N) \geq \mathrm{cl}(\xi) - 1. \tag{2-11}$$

Together with (2-2) this will complete the proof of the Theorem. The proof of (2-11) will consist of building a *polynomial deformation* of the cochain complex $C^*(X;\mathcal{E}_a)$ (where a is viewed as a parameter) into $C^*(N,\partial_+N)$ as $a\to\infty$. The *deformation* understood here as a finitely generated free cochain complex C^* over the ring $P=\mathbf{Z}[\tau]$ of polynomials with integral coefficients satisfying (a) and (b) below.

The construction of the deformation C^* goes as follows. We shall assume that N is triangulated and ∂N is a subcomplex. Recall that $V=f^{-1}(1)$ (cf. 2.1) and we will denote by $i_\pm : V\to N$ the inclusions, which identify V with $\partial_\pm N$ correspondingly. Denote by $C^q(N)$ and $C^q(V)$ the free abelian groups of integer valued cochains and $\delta_N : C^q(N)\to C^{q+1}(N)$ and by $\delta_V : C^q(V)\to C^{q+1}(V)$ the coboundary homomorphisms.

Let $C^q(N)[\tau]$ and $C^{q-1}(V)[\tau]$ denote free P-modules formed by "polynomials with coefficients" in the corresponding abelian groups; for example, an element $c\in C^q(N)[\tau]$ is a formal sum $c=\sum_{i\geq 0} c_i\tau^i$ with $c_i\in C^q(N)$ and only finitely many c_i's are nonzero. We shall consider of $c\in C^q(N)[\tau]$ as a polynomial (complex) curve, which associates a point in $C^q(N)$ with a complex number $\tau\in\mathbf{C}$. The P-module structure is given naturally as follows: $\tau\cdot c=\sum_{i\geq 0} c_i\tau^{i+1}$. It is clear that $C^q(N)[\tau]$ and $C^{q-1}(V)[\tau]$ are free finitely generated P-modules and their ranks equal to the number of q-dimensional simplices in N or $(q-1)$-dimensional simplices in V, correspondingly.

Consider the natural P-module extensions

$$\delta_N : C^q(N)[\tau]\to C^{q+1}(N)[\tau], \quad \text{and} \quad \delta_V : C^q(V)[\tau]\to C^{q+1}(V)[\tau]. \tag{2-12}$$

They act coefficientwise so that δ_N and δ_V are P-homomorphisms. For example, if $\alpha=\sum_{i\geq 0}\alpha_i\tau^i\in C^q(N)[\tau]$ then $\delta_N(\alpha)=\sum_{i\geq 0}\delta_N(\alpha_i)\tau^i$.

Now we define a finitely generated free cochain complex C^* over the ring $P=\mathbf{Z}[\tau]$ of polynomials with integral coefficients as follows: $C^*=\oplus C^q$, where

$$C^q = C^q(N)[\tau]\oplus C^{q-1}(V)[\tau]. \tag{2-13}$$

Elements of C^q will be denoted as pairs (α,β), where $\alpha\in C^q(N)[\tau]$ and $\beta\in C^{q-1}(V)[\tau]$. The differential $\delta : C^q\to C^{q+1}$ is given by the following formula:

$$\delta(\alpha,\beta) = (\delta_N(\alpha), (i_+^* - \tau i_-^*)(\alpha) - \delta_V(\beta)), \tag{2-14}$$

where $\alpha \in C^q(N)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. It is clear that C^* is the usual cylinder of the chain map $i_+^* - \tau i_-^*$ with a shifted grading.

We claim now that:

(a) for any nonzero complex number $a \in \mathbf{C}^*$ there is a canonical isomorphism

$$E_a^q : H^q(C^* \otimes_P \mathbf{C}_a) \xrightarrow{\simeq} H^q(X; \mathcal{E}_{a^{-1}}). \tag{2-15}$$

Here $\mathbf{C}_a$ is $\mathbf{C}$ which is viewed as a P-module with the following structure: $\tau x = ax$ for $x \in \mathbf{C}$. We will call E_a^q *an isomomorphism of evaluation at $\tau = a$;*

(b) *for $a = 0$ we also have a canonical evaluation isomorphism*

$$E_0^q : H^q(C^* \otimes_P \mathbf{Z}_0) \to H^q(N, \partial_+ N; \mathbf{Z}), \tag{2-16}$$

where $\mathbf{Z}_0$ is $\mathbf{Z}$ with the following P-module structure: $\tau x = 0$ for any $x \in \mathbf{Z}$.

To show (a) we note that $H^q(X; \mathcal{E}_{a^{-1}})$ can be identified with the cohomology of complex $C^*(X; \mathcal{E}_{a^{-1}})$ consisting of cochains $\alpha \in C^q(N)$ satisfying the boundary conditions

$$i_-^*(\alpha) = a^{-1} i_+^*(\alpha) \in C^q(V).$$

The complex $C^* \otimes_P \mathbf{C}_a$ can be viewed as

$$C^q \otimes_P \mathbf{C}_a = C^q(N) \oplus C^{q-1}(V)$$

with the differential given by

$$\delta(\alpha, \beta) = (\delta_N(\alpha), (i_+^* - a i_-^*)(\alpha) - \delta_V(\beta)), \tag{2-17}$$

where $\alpha \in C^q(N)$ and $\beta \in C^{q-1}(V)$. It is clear that there is a chain homomorphism $C^*(X; \mathcal{E}_{a^{-1}}) \to C^* \otimes_P \mathbf{C}_a$ (acting by $\alpha \mapsto (\alpha, 0)$). It is easy to see that it induces an isomorphism on the cohomology. Indeed, suppose that a cocycle $\alpha \in C^q(X; \mathcal{E}_{a^{-1}})$ bounds in the complex $C^* \otimes_P \mathbf{C}_a$ then there are $\alpha_1 \in C^{q-1}(N)$, $\beta_1 \in C^{q-2}(V)$ such that $\alpha = \delta_N(\alpha_1)$, $i_+^*(\alpha_1) - a i_-^*(\alpha_1) - \delta_V(\beta_1) = 0$. We may find a cochain $\beta_2 \in C^{q-2}(N)$ such that $i_+^*(\beta_2) = \beta_1$ and $i_-^*(\beta_2) = 0$ (by extending β_1 into a neighborhood of $\partial_+ N$). Then setting $\alpha_2 = \alpha_1 - \delta_N(\beta_2)$ we have

$$\alpha = \delta_N(\alpha_2), \qquad i_+^*(\alpha_2) - a i_-^*(\alpha_2) = 0, \tag{2-18}$$

which means that α also bounds in $C^q(X; \mathcal{E}_{a^{-1}})$.

Similarly, suppose that (α, β) is a cocycle of complex $C^* \otimes_P \mathbf{C}_a$. As above we may find a cochain $\beta' \in C^{q-1}(N)$ with $i_+^*(\beta') = \beta$ and $i_-^*(\beta') = 0$. Then $(\alpha - \delta_N(\beta'), 0)$ is cohomologous to the initial cocycle (α, β) and it is a cocycle of $C^*(X; \mathcal{E}_{a^{-1}})$.

This proves (a). The statement (b) follows similarly. $\square$

2.4. Relative deformation complex. We will define now a relative version of the deformation complex C^*.

Let $A \subset N$ be a simplicial subcomplex. We will assume that A *is disjoint from* $\partial_+ N$. Let $C^q(N, A)$ denote the free abelian group of integer-valued cochains on N

which vanish on A. Let $C^q(N,A)[\tau]$ be constructed similarly to $C^q(N)[\tau]$, cf. above. We define the complex C^*_A as follows:

$$C^q_A = C^q(N,A)[\tau] \oplus C^{q-1}(V)[\tau]. \tag{2-19}$$

The differential $\delta : C^q_A \to C^{q+1}_A$ is defined by the following formula:

$$\delta(\alpha,\beta) = (\delta_{N,A}(\alpha), (i^*_+ - \tau i^*_-)(\alpha) - \delta_V(\beta)), \tag{2-20}$$

where $\alpha \in C^q(N,A)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. Here $\delta_{N,A} : C^q(N,A) \to C^{q+1}(N,A)$ and $\delta_V : C^q(V) \to C^{q+1}(V)$ denote the coboundary homomorphisms and also their P-module extension. $i^*_\pm : C^q(N,A) \to C^q(V)$ denote the restriction maps of chains, and the same symbols denote also their polynomial extensions $i^*_\pm : C^q(N,A)[\tau] \to C^q(V)[\tau]$.

Similarly to statements (a) and (b) in 2.3 we have:

(a') for any $a \in \mathbf{C}^*$ there is a natural isomorphism

$$H^i(C^*_A \otimes_P \mathbf{C}_a) \simeq H^i(X, p(A); \mathcal{E}_{a^{-1}}), \tag{2-21}$$

where $p : N \to X$ is the identification map, cf. 2.1;

(b') also,

$$H^i(C^*_A \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \mathbf{Z}). \tag{2-22}$$

2.5. Algebraic integers and the lifting property. Here it will become clear why our definition of the cup-length $\mathrm{cl}(\xi)$ involves the condition of not being a Dirichlet unit.

Proposition 2. *Suppose that $A \subset N$ is a subcomplex such that the inclusion $A \to N$ is homotopic to a map $A \to \partial_+ N$ keeping $A \cap \partial_+ N$ fixed. Let $a \in \mathbf{C}^*$ be a complex number such that a^{-1} is not an algebraic integer. Then the homomorphism $C^*_A \to C^*$ induces an epimorphism on the cohomology*

$$H^i(C^*_A \otimes_P \mathbf{C}_a) \to H^i(C^* \otimes_P \mathbf{C}_a), \qquad i = 0, 1, 2, \dots. \tag{2-23}$$

Proof. Let $\mathbf{Z}_0$ denote the group $\mathbf{Z}$ considered as a P-module with the trivial τ action, i.e. $\mathbf{Z}_0 = P/\tau P$. We will show first that

$$H^i(C^*_A \otimes_P \mathbf{Z}_0) \to H^i(C^* \otimes_P \mathbf{Z}_0) \tag{2-24}$$

is an epimorphism. We know from (b') of subsection 2.4 that

$$H^i(C^*_A \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \mathbf{Z}) \quad \text{and} \quad H^i(C^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, \partial_+ N; \mathbf{Z}).$$

In the exact sequence

$$\dots \to H^i(N, A \cup \partial_+ N; \mathbf{Z}) \to H^i(N, \partial_+ N; \mathbf{Z}) \xrightarrow{j^*} H^i(A \cup \partial_+ N, \partial_+ N; \mathbf{Z}) \to \dots$$

j^* acts trivially (since the inclusion $(A \cup \partial_+ N, \partial_+ N) \to (N, \partial_+ N)$ is null-homotopic) and hence $H^i(N, A \cup \partial_+ N; \mathbf{Z}) \to H^i(N, \partial_+ N; \mathbf{Z})$ is an epimorphism. This proves that (2-24) is an epimorphism. Now, Proposition 2 follows from Lemma 5 below. $\square$

Lemma 5. *Let C and D be chain complexes of free finitely generated $P = \mathbf{Z}[\tau]$-modules and let $f : C \to D$ be a chain map. Suppose that for some q the induced map $f_* : H_q(C \otimes_P \mathbf{Z}_0) \to H_q(D \otimes_P \mathbf{Z}_0)$ is an epimorphism; here $\mathbf{Z}_0$ is $\mathbf{Z}$ considered with the trivial P-action: $\mathbf{Z}_0 = P/\tau P$. Then for any complex number $a \in \mathbf{C}^*$, such that a^{-1} is not an algebraic integer, the homomorphism*

$$f_* : H_q(C \otimes_P \mathbf{C}_a) \to H_q(D \otimes_P \mathbf{C}_a) \tag{2-25}$$

is an epimorphism; here $\mathbf{C}_a$ denotes $\mathbf{C}$ with τ acting as the multiplication by a.

Proof. Denote by $Z_q(C), Z_q(D)$ the cycles of C and D and by $B_q(C)$ and $B_q(D)$ their boundaries. Recall that the homological dimension of P is 2. We have the exact sequence

$$0 \to Z_q(C) \to C_q \to B_{q-1}(C) \to 0$$

and hence $Z_q(C)$ is a free P-module (since $B_{q-1}(C)$ is a submodule of a free module and so has a homological dimension ≤ 1). Similarly $Z_q(D)$ is free.

Choose a basis for $Z_q(C), Z_q(D)$ and D_{q+1} and express in terms of these basis the map

$$f \oplus d : Z_q(C) \oplus D_{q+1} \to Z_q(D). \tag{2-26}$$

The resulting matrix M is a rectangular matrix with entries in P.

We claim: *there exist integers $b_j \in \mathbf{Z}$ and minors $A_j(\tau) \in P$ of the matrix M of size* $\operatorname{rk} Z_q(D) \times \operatorname{rk} Z_q(D)$, *such that the polynomial with integer coefficients*

$$p(\tau) = \sum_j b_j A_j(\tau) \tag{2-27}$$

satisfies

$$p(0) = 1. \tag{2-28}$$

In fact, we will show that our claim is *equivalent* to the requirement that $f_* : H_q(C \otimes_P \mathbf{Z}_0) \to H_q(D \otimes_P \mathbf{Z}_0)$ is an isomorphism. Namely, using the resolvent $0 \to P \xrightarrow{\tau} P \to \mathbf{Z}_0 \to 0$ it is easy to see that $\operatorname{Tor}_1^P(B_{q-1}(C), \mathbf{Z}_0) = 0$ (since $B_{q-1}(C)$ is a submodule of a free module). Hence we have the exact sequence

$$0 \to Z_q(C) \otimes_P \mathbf{Z}_0 \to C_q \otimes_P \mathbf{Z}_0 \to B_{q-1}(C) \otimes \mathbf{Z}_0 \to 0.$$

This means that $Z_q(C) \otimes_P \mathbf{Z}_0 = Z_q(C \otimes_P \mathbf{Z}_0)$, and $B_{q-1}(C) \otimes_P \mathbf{Z}_0 = B_{q-1}(C \otimes_P \mathbf{Z}_0)$. Hence, the hypothesis of the lemma implies that the homomorphism

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{Z}_0) \oplus (D_{q+1} \otimes_P \mathbf{Z}_0) \to Z_q(D) \otimes_P \mathbf{Z}_0$$

is an epimorphism. This epimorphism is described by the matrix $M(0)$, where we substitute $\tau = 0$ into M. Therefore, there are minors $A_j(\tau)$ of M of size $\operatorname{rk} Z_q(D) \times \operatorname{rk} Z_q(D)$ so that the ideal in $\mathbf{Z}$ generated by the integers $A_j(0)$ contains 1. This proves (2-28).

Since $p(\tau)$ is an integral polynomial with $p(0) = 1$ and a^{-1} is not an algebraic integer it follows that

$$p(a) \neq 0. \tag{2-29}$$

Let us show that (2-29) is equivalent to the statement that (2-25) is an epimorphism. We have the exact sequence

$$0 \to Z_q(C) \otimes_P \mathbf{C}_a \to C_q \otimes_P \mathbf{C}_a \to B_{q-1} \otimes \mathbf{C}_a \to 0$$

(here we may work over $\mathbf{C}[\tau]$ which is a PID). Hence, similarly to the arguments above, we obtain that the map

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{C}_a) \oplus (D_{q+1} \otimes_P \mathbf{C}_a) \to Z_q(D) \otimes_P \mathbf{C}_a \tag{2-30}$$

is described by the matrix M with substitution $\tau = a$. We conclude that at least one of the $\operatorname{rk} Z_q(D) \times \operatorname{rk} Z_q(D)$ minors $A_j(a)$ is nonzero because of (2-29), and hence (2-30) and (2-25) are epimorphisms. $\square$

2.6. Corollary. *Let $a \in \mathbf{C}^*$ be a complex number, not an algebraic integer. Let $A \subset X$ be a closed subset such that $A = p(A')$, where $A' \subset N - \partial_+ N$ is a closed polyhedral subset such that the inclusion $A' \to N$ is homotopic to a map with values in $\partial_+ N$. Then the restriction map*

$$H^q(X, A; \mathcal{E}_a) \to H^q(X; \mathcal{E}_a) \tag{2-31}$$

is an epimorphism.

Proof. We just combine the isomorphisms (a) and (a') (cf. 2.3, 2.4) and Proposition 2. $\square$

2.7. End of proof of Theorem 1. We need to establish inequality (2-11). In other words, we want to prove the triviality of any cup-product

$$v_0 \cup v_1 \cup \dots \cup v_{m+1} = 0, \quad \text{where} \quad v_j \in H^{d_j}(X; \mathcal{E}_{a_j}), \quad d_j > 0, \tag{2-32}$$

assuming that among the numbers $a_0, a_1, \dots, a_{m+1} \in \mathbf{C}$ at least two are not Dirichlet units; here m denotes $m = \operatorname{cat}_{(N, \partial_+ N)}(N)$.

We shall assume that a_0 and a_{m+1} are not Dirichlet units; if not, we just rename the numbers.

Moreover, we will assume that one of the numbers a_0 and a_{m+1} is not an algebraic integer. In the case when both a_0 and a_{m+1} are algebraic integers the inverse numbers a_0^{-1} and $a_{m+1}^{-1}$are not algebraic integers and we shall apply the arguments following below to the form $-\omega$ (representing the cohomology class $-\xi$), which obviously has the same set of critical points.)

Since we may always rename the numbers a_0 and a_{m+1}, we will assume below that a_0 is not an algebraic integer.

Suppose that N can be covered by closed subsets $A_0, A_1 \cup \dots \cup A_m = N$ so that A_0 is a collar of $\partial_+ N$ (cf. 2.2), and for $j = 1, 2, \dots, m$ the subset A_j is disjoint from $\partial_+ N$ and null-homotopic in N. Using Lemma 3, we may assume that the sets A_j are polyhedral. We find (since $d_j > 0$) that for $j = 1, 2, \dots, m$ we may lift the class v_j to a relative cohomology class $\tilde{v}_j \in H^{d_j}(X, B_j; \mathcal{E}_{a_j})$, where $B_j = p(A_j)$.

Let $U_\pm$ be a small cylindrical neighborhood of $\partial_\pm N$ in N. Applying Corollary 2.6, class v_0 can be lifted to a class $\tilde{v}_0 \in H^{d_j}(X, B_0; \mathcal{E}_{a_0})$, where $B_0 = p(A_0 - U_+)$.

Let B_{m+1} be a closed cylindrical neighborhood of V in X containing $\overline{p(U_-)}\cup\overline{p(U_+)}$. We claim that we may lift the class $v_{m+1} \in H^{d_{m+1}}(X;\mathcal{E}_{a_{m+1}})$ to a class $\tilde{v}_{m+1} \in H^{d_{m+1}}(X, B_{m+1};\mathcal{E}_{a_{m+1}})$. We will use Corollary 2.6. First, find two shifts of V into $X - B_{m+1}$, one (denoted V') in the positive normal direction and the other (denoted V'') in the negative normal direction (cf. Figure 5). If the number a_{m+1} is not an algebraic integer we may apply Corollary 2.6 to the cut V''. If the number a_{m+1}^{-1} is not an algebraic integer we may apply Corollary 2.6 to the cut V'.

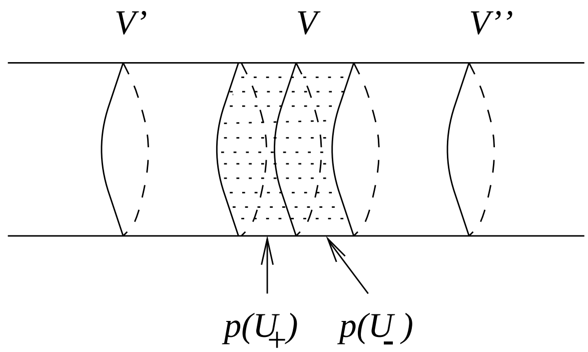

Figure 5.

Now, it is clear that the product $v_0\cup\dots\cup v_{m+1}$ is trivial since it is obtained from the product $\tilde{v}_0\cup\dots\cup\tilde{v}_{m+1}$ (lying in $H^d(X,\cup_{j=0}^{m+1}B_j;\mathcal{E}_a)$, where $a = a_0a_1\dots a_{m+1}$) by restricting onto X, and the group $H^d(X,\cup_{j=0}^{m+1}B_j;\mathcal{E}_a)$ vanishes since $X = \cup_{j=0}^{m+1}B_j$. $\square$

§3. Proofs of Theorems 2 and 3 and Proposition 1

3.1. Proof of Theorem 2. Consider the deformation complex constructed in subsection 2.3. It is a free finitely generated cochain complex $C = \oplus C^q$ over the ring $P = \mathbf{Z}[\tau]$, which satisfies (2-15). We denote by $B^q(a)$ the rank of the following linear map

$$\delta\otimes 1 : C^{q-1}\otimes_P \mathbf{C}_a \to C^q\otimes_P \mathbf{C}_a. \tag{3-1}$$

It is clear that this map $\delta\otimes 1$ can be represented by a square matrix whose entries are polynomials in a with integral coefficients. Therefore we obtain that $B^q(a)$ *as a function of a is constant except possibly at finitely many jump points $a = a_1^q,\dots,a_k^q$, which are all algebraic numbers, and at those jump values $a = a_1^q,\dots,a_k^q$ the value of $B^q(a)$ is smaller than at a generic point.* Applying the Euler - Poincaré formula to the truncated complex $0\to C^q\to C^{q+1}\to\dots$ and also (2-15) we find

$$\begin{aligned}\dim H^q(X;\mathcal{E}_{a^{-1}}) = &\sum_{i=0}^{\infty}(-1)^i \operatorname{rk} C^{q+i} - \\ &-\sum_{j=1}^{\infty}(-1)^j \dim H^{q+j}(X;\mathcal{E}_{a^{-1}}) - B^q(a).\end{aligned} \tag{3-2}$$

The sums here are actually finite. This proves statements (1), (3), (4) of Theorem 2. Statement (2) follows from the definition of the Novikov numbers given in [F], subsection 1.2. According to this definition, the number $b_i(\xi)$ is the dimension over the field of rational functions $F(\tau)$ of the local system of rank 1 over X which is

naturally determined by the class ξ, cf. [F], section 1.2. Therefore, it is obvious that $b_i(\xi)$ coincides with the value $\dim H^i(X, \mathcal{E}_a)$ for a generic point $a \in \mathbf{C}^*$. In section 1.4 of [F] it is shown that this definition of $b_i(\xi)$ (using the rational functions) is in fact equivalent to the original definition of S.P. Novikov using the formal power series. $\square$

3.2. Proof of Theorem 3. Let ω be a closed 1-form lying in a cohomology class $\xi \in H^1(X; \mathbf{R})$ of rank $= r > 1$. Let $S = S(\omega)$ denote the set of zeros of ω. It is clear that $\xi|_S = 0$.

Let $\xi_1, \dots, \xi_r \in H^1(X; \mathbf{Z})$ be a basis of the free abelian group $\mathrm{Hom}(H_1(X)/\ker(\xi))$, where r is the rank of ξ. We may write $\xi = \sum_{i=1}^r \alpha_i \xi_i$, and the coefficients are real $\alpha_i \in \mathbf{R}$.

Suppose that ξ_m is a sequence of rank 1 classes with $\mathrm{cl}(\xi_m) \geq \mathrm{cl}(\xi)$, which converges to ξ as $m \to \infty$, and each of the classes ξ_m vanishes on $\ker(\xi)$. Then we have $\xi_m = \sum_i \alpha_{i,m}\xi_i$, where $\alpha_{i,m} = \lambda_m \cdot n_{i,m}$, $\lambda_m \in \mathbf{R}$, and $n_{i,m} \in \mathbf{Z}$ for $i = 1, 2, \dots, r$. Each sequence $\alpha_{i,m}$ converges to α_i as m tends to ∞.

Choose a closed 1-form ω_i in the class ξ_i for $i = 1, \dots, r$; since $\xi_i|_S = 0$ we may choose it so that it vanishes identically on a neighborhood of S. Define the following sequence of closed 1-forms

$$\omega_m = \omega - \sum_{i=1}^r (\alpha_i - \alpha_{i,m})\omega_i.$$

It is clear that ω_m has rank 1 and for m large enough $S(\omega_m) = S(\omega)$. The cohomology class of ω_m is ξ_m. By Theorem 1 we have $\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(\xi_m) - 1$. Hence we obtain $\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(\xi) - 1$. $\square$

3.3. Proof of Proposition 1. It is clear that it is enough to prove (1-10) assuming that the classes ξ_1 and ξ_2 are integral $\xi_\nu \in H^1(X_\nu; \mathbf{Z})$ for $\nu = 1, 2$. The general statement then follows automatically due to the nature of our definition of $\mathrm{cl}(\xi)$ for general ξ, cf. 1.6.

Position X_1 and X_2 so that their intersection is a small n-dimensional disk D^n, and then the connected sum $X_1 \# X_2$ is obtained from the union $X_1 \cup X_2$ by removing the interior of D^n. Let $\mathcal{E}$ be a flat bundle over the connected sum $X_1 \# X_2$ and let $\mathcal{E}_\nu$ be a flat bundle over X_ν so that

$$\mathcal{E}|_{X_\nu - \overset{\circ}{D^n}} \simeq \mathcal{E}_\nu|_{X_\nu - \overset{\circ}{D^n}}, \tag{3-3}$$

for $\nu = 1, 2$. As follows from the Mayer - Vietoris sequence there is a canonical isomorphism

$$\psi : H^q(X_1; \mathcal{E}_1) \oplus H^q(X_2; \mathcal{E}_2) \to H^q(X_1 \# X_2; \mathcal{E}) \tag{3-4}$$

for $0 < q < n = \dim X_1 = \dim X_2$. For $q = n$ the homomorphism ψ is only an epimorphism, but its restriction on any of the summands $H^n(X_1; \mathcal{E}|_{X_1})$ or $H^n(X_2; \mathcal{E}|_{X_2})$ is an isomorphism.

Also, ψ is multiplicative in the following sense. Suppose that we have another flat bundle $\mathcal{F}$ over the connected sum $X_1 \# X_2$ and let $\mathcal{F}_\nu$ be flat bundles over X_ν, $\nu = 1, 2$ satisfying condition (3-3). Then for any $v \in H^*(X_1, \mathcal{E}_1)$ and $w \in H^*(X_1, \mathcal{F}_1)$ holds $\psi(v \cup w, 0) = \psi(v, 0) \cup \psi(w, 0)$ and similarly with respect to the other variable.

Given a complex number $a \in \mathbf{C}^*$, it determines (as in 1.2) the flat line bundles $\mathcal{E}_\nu$ over X_ν (together with the class ξ_ν, $\nu = 1, 2$) and a flat line bundle $\mathcal{E}$ over $X = X_1 \# X_2$ (together with $\xi = \xi_1 \# \xi_2$). These three flat bundles clearly satisfy (3-3).

Suppose now that we have classes $v_j \in H^{d_j}(X_1; \mathcal{E}_{a_j})$, where $j = 1, 2, \dots, k$ such that their product $v_1 \cup \dots \cup v_k$ is non-trivial. We assume that at least two a_j are not Dirichlet units, and this implies that $d_j < n$ for all j. Then we obtain classes $w_j = \psi(v_j, 0) \in H^{d_j}(X; \mathcal{E}_{a_j})$ with $w_1 \cup \dots \cup w_k \neq 0$. This proves inequality $\mathrm{cl}(\xi) \geq \mathrm{cl}(\xi_1)$. Therefore $\mathrm{cl}(\xi) \geq \max\{\mathrm{cl}(\xi_1), \mathrm{cl}(\xi_2)\}$.

The inverse inequality follows similarly, using the properties of the map ψ mentioned above. $\square$

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